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FAST TRACK COMMUNICATION

A stabilizer code for uncorrelated errors can correct spatially correlated ones

T Hayashi[1,3], T Arimitsu[1], S Kitajima[1] and F Shibata[2]

[1] Graduate School of Pure and Applied Sciences, University of Tsukuba, Ibaraki 305-8571, Japan
[2] Department of Physics, Faculty of Science, Ochanomizu University, Tokyo 112-8610, Japan

E-mail: tarimtsu@sakura.cc.tsukuba.ac.jp

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Abstract

It is shown that errors on qubits caused by *spatially correlated* noise of quantum Brownian motion can be corrected with a stabilizer code (a quantum error correction code) and recovery operations prepared for *uncorrelated* noise without modifications. The analysis is made by means of the quantum stochastic Liouville equation approach which has been developed within the canonical operator formalism for dissipative systems called non-equilibrium thermo field dynamics. This approach yields a transparent procedure to derive completely positive maps describing errors on qubits under given statistics of noise.

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Any intrinsic interaction between a relevant quantum system and its environment causes inevitable noise which corrupts the quantum state of the relevant system in a stochastic manner. The noise is considered to be one of major obstacles which will hamper large-scale practical quantum information processing (QIP) including quantum computation, superdense coding, quantum cryptography, etc [1]. Without any technique of noise reduction or error correction, efficiency of QIP will be diminished. In order to overcome the influence of noise, there have been developed quantum error correction code (QECC), mainly the stabilizer code [2–6], and other methods of noise reduction, e.g., that utilizing the decoherence-free subspace [7–9].

The existing QECCs are designed, mainly, for the cases where each qubit suffers from noise which is spatially and temporally independent [1, 2, 10, 11], i.e., the noise affecting one qubit is statistically independent of that affecting other qubits (spatial independence) and has no temporal correlation among them (temporal independence). Note that qubits on different sites are mutually entangled quantum-mechanically, in general. The neglection of spatial correlations of noise can be a good approximation when the distance between adjacent

[3] Present address: MCC Corporation, Bunkyo-ku, Tokyo 112-0002, Japan.

qubits is large compared to the correlation length of noise. Although the neglection of spatial and/or temporal correlations of noise allows one to create a mathematical noise model without considering physical detail about qubit-environment interactions, the validity of the neglection is not evident in practical QIPs [2]. For instance, qubits embodied by nuclear spins [12–14] can be under the strong influence of spatially correlated noise when the external magnetic field has fluctuation of wavelength comparable to the spacing between nuclear spins [15]. Therefore, it is desirable to understand how well the existing QECCs function in such realistic situations where noises are correlated.

In this paper, we prove that errors caused by *spatially correlated* noise can be corrected by the stabilizer code and error correction operation prepared for *uncorrelated* ones, and thus, the neglection of spatial correlation is justified. Here, we characterize the noise by *quantum Brownian motion* [16–21] which represents zero-point fluctuation in addition to thermal fluctuation. The time-evolution of both qubits and noise systems is described by the stochastic Liouville equation [16, 17, 22] within *non-equilibrium thermo field dynamics* (NETFD) [23–25] which is canonical operator formalism for open systems providing us with transparent and powerful methods to tackle problems in quantum information theory, especially, those related to decoherence and/or dissipation. The completely positive map representing errors on qubits is given by the time-evolution operator $\hat{\mathcal{E}}(t)$ introduced in (5).

Within NETFD, any operator A is accompanied by its *tilde* counterpart $\tilde{A}$ which is created by the *tilde conjugation*, $(\bullet)^{\sim}$, as $(A)^{\sim} = \tilde{A}$, $(\tilde{A})^{\sim} = A$ [22, 26]. The tilde conjugation is defined by $(c_1\mathcal{O}_1+c_2\mathcal{O}_2)^{\sim} = c_1^*(\mathcal{O}_1)^{\sim}+c_2^*(\mathcal{O}_2)^{\sim}$, $(\mathcal{O}_1\mathcal{O}_2)^{\sim} = (\mathcal{O}_1)^{\sim}(\mathcal{O}_2)^{\sim}$, $(\mathcal{O}^{\dagger})^{\sim} = ((\mathcal{O})^{\sim})^{\dagger}$. The operator $\mathcal{O}_\bullet$ is an arbitrary operator acting on the doubled Hilbert space $\hat{\mathsf{V}} = \mathsf{V}\otimes\tilde{\mathsf{V}}$, where V is the Hilbert space for quantum field theory or quantum mechanics, and $\tilde{\mathsf{V}}$ has the same structure as V. Here, $c_\bullet$ is a c-number.

The statistical average $\langle\mathcal{O}(t)\rangle$ is given by the vacuum expectation value $\langle\mathcal{O}(t)\rangle = \langle\!\langle\theta|\mathcal{O}|0(t)\rangle\!\rangle$, where $|0(t)\rangle\!\rangle$ and $\langle\!\langle\theta|$ are, respectively, the *thermal ket- and bra-vacuum* normalized as $\langle\!\langle\theta|0(t)\rangle\!\rangle = \langle 1\rangle = 1$. The ket-vacuum belongs to $\hat{\mathsf{V}}$, and the bra-vacuum to $\hat{\mathsf{V}}^{\dagger}$. Note that $\mathcal{O}_\bullet$ acts also on $\hat{\mathsf{V}}^{\dagger}$. The observable operator Q is defined only by *non-tilde* operators which act on $\hat{\mathsf{V}}$ and $\hat{\mathsf{V}}^{\dagger}$. The thermal vacua are required to be tilde-invariant, i.e., $\langle\!\langle\theta|^{\sim} = \langle\!\langle\theta|$, $|0(t)\rangle\!\rangle^{\sim} = |0(t)\rangle\!\rangle$. The tilde-invariance of thermal vacua guarantees that the statistical average of an observable operator $\langle Q(t)\rangle$ is a real number. Irreversible dynamics of the system is described by the time-dependent ket-vacuum $|0(t)\rangle\!\rangle$.

Let us consider a system that consists of n qubits (a relevant system) and spatially correlated quantum noise (an irrelevant system). We assume that the dynamics of the total system is described by the stochastic Liouville equation

$$d|0_{\mathrm{tot}}(t)\rangle\!\rangle = -\mathrm{i}\hat{\mathcal{H}}_t\,\mathrm{d}t\circ|0_{\mathrm{tot}}(t)\rangle\!\rangle \tag{1}$$

of the Stratonovich type for the total ket-vacuum $|0_{\mathrm{tot}}(t)\rangle\!\rangle$ within an interaction representation. Here, the symbol '$\circ$' denotes the Stratonovich stochastic product[4], and the hat-Hamiltonian $\hat{\mathcal{H}}_t\,\mathrm{d}t = \mathcal{H}_t\,\mathrm{d}t - \tilde{\mathcal{H}}_t\,\mathrm{d}t$ with

$$\mathcal{H}_t\,\mathrm{d}t = -\sum_{i=1}^{n}\left(\mathrm{d}\mathfrak{B}_t^{X_i}X_i + \mathrm{d}\mathfrak{B}_t^{Y_i}Y_i + \mathrm{d}\mathfrak{B}_t^{Z_i}Z_i\right) \tag{2}$$

describing the interaction between the relevant and irrelevant systems. X_i, Y_i, Z_i are the Pauli operators for the ith qubit. $\mathrm{d}\mathfrak{B}_t^{\bullet}\left(=\mathrm{d}\mathfrak{B}_t^{\bullet\dagger}\right)$ is the quantum Brownian motion specified by the

[4] The Stratonovich stochastic product is related to the Itô product '$\cdot$' by $X_t\circ\mathrm{d}Y_t = X_t\cdot\mathrm{d}Y_t + \frac{1}{2}\,\mathrm{d}X_t\mathrm{d}Y_t$. Here, X_t and Y_t are the stochastic operators. The increment $\mathrm{d}Y_t \equiv Y_{t+\mathrm{d}t} - Y_t$ is of $O(\sqrt{\mathrm{d}t})$. Note that the random average of the Itô product satisfies $\langle X_t\cdot\mathrm{d}Y_t\rangle = \langle X_t\rangle\langle\mathrm{d}Y_t\rangle$.

first and second moments

$$\langle\langle|\mathrm{d}\mathfrak{B}_t^\bullet|\rangle\rangle = 0, \qquad \langle\langle|\mathrm{d}\mathfrak{B}_t^\alpha\,\mathrm{d}\mathfrak{B}_t^\beta|\rangle\rangle = D_{\alpha\beta}\,\mathrm{d}t, \tag{3}$$

where $\langle\langle|$ and $|\rangle\rangle$ are, respectively, the bra- and ket-vacua for the irrelevant system. The vacuum expectation $\langle\langle|\cdots|\rangle\rangle$ can be understood as a random average. Note that $|\rangle\rangle \neq \langle\langle|^\dagger$. $\mathrm{d}\mathfrak{B}_t^\bullet$ and $\mathrm{d}\tilde{\mathfrak{B}}_t^\bullet$ satisfy the *thermal state condition* (TSC) [23–25] $\langle\langle|\mathrm{d}\tilde{\mathfrak{B}}_t^\bullet = \langle\langle|\mathrm{d}\mathfrak{B}_t^\bullet$. The superscripts α and β in (3) stand for an element out of $\mathcal{G}^{(1)} \equiv \{X_i, Y_i, Z_i\}_{i=1}^n$. The correlation matrix D is a positive matrix characterizing physical nature of the noise. Since noises are spatially correlated in general, $D_{\alpha_i\beta_j} \neq 0$ for $i \neq j$, where α_i and β_j stand for, respectively, one out of $\{X_i, Y_i, Z_i\}$ and $\{X_j, Y_j, Z_j\}$. The bra-vacuum $\langle\langle\theta| \otimes \langle\langle|$ for the total system satisfies $\langle\langle\theta| \otimes \langle\langle|\hat{\mathcal{H}}_t\,\mathrm{d}t = 0$ which is a manifestation of the conservation of probability with respect to the total system. The hat-Hamiltonian is *tildian* satisfying $(\mathrm{i}\hat{\mathcal{H}}_t\,\mathrm{d}t)^\sim = \mathrm{i}\hat{\mathcal{H}}_t\,\mathrm{d}t$ which guarantees the tilde-invariance of vacua. The bra-vacuum $\langle\langle\theta|$ for qubits satisfies the TSC $\langle\langle\theta|\tilde{A} = \langle\langle\theta|A^\dagger$ with A being any operator for the relevant system.

We adopt here the initial condition $|0_{\mathrm{tot}}(0)\rangle\rangle = |0\rangle\rangle \otimes |\rangle\rangle$ [17] for (1). Here, $|0\rangle\rangle$ is the initial ket-vacuum of the relevant system.

By taking the random average of (1), we obtain the quantum master equation for the ket-vacuum $|0(t)\rangle\rangle \equiv \langle\langle|0_{\mathrm{tot}}(t)\rangle\rangle$ of the relevant system in the form

$$\frac{\partial}{\partial t}|0(t)\rangle\rangle = \hat{\Pi}|0(t)\rangle\rangle \tag{4}$$

with $\hat{\Pi}\,\mathrm{d}t \equiv -\frac{1}{2}\langle\langle|(\hat{\mathcal{H}}_t\,\mathrm{d}t)^2|\rangle\rangle = \frac{1}{2}\sum_{\alpha,\beta\in\mathcal{G}^{(1)}} D_{\alpha\beta}^*\hat{\Pi}_{\alpha\beta}\,\mathrm{d}t$, where $\hat{\Pi}_{\alpha\beta} \equiv 2\alpha\tilde{\beta} - \beta\alpha - \tilde{\alpha}\tilde{\beta}$. In deriving (4), we have used the TSC and (3). Note that the generator $\hat{\Pi}$ has direct correspondence to Lindblad's generator [27]. Integrating (4), we obtain $|0(t)\rangle\rangle = \hat{\mathcal{E}}(t)|0\rangle\rangle$ with the correlated error operator

$$\hat{\mathcal{E}}(t) = \mathrm{e}^{t\hat{\Pi}} \tag{5}$$

which is a completely positive map.

When the adjacent qubits are well separated, the noise affecting each qubit becomes statistically independent, i.e., $D_{\alpha_i\beta_j} \propto \delta_{ij}$. In a more extreme case where $\mathrm{d}\mathfrak{B}_t^{X_i}$, $\mathrm{d}\mathfrak{B}_t^{Y_i}$ and $\mathrm{d}\mathfrak{B}_t^{Z_i}$ are mutually independent, we have $D_{\alpha\beta} = D_{\alpha\alpha}\delta_{\alpha\beta}$. With this 'fully independent' correlation matrix, $\hat{\mathcal{E}}(t)$ is reduced to

$$\hat{\mathcal{E}}_0(t) = \mathrm{e}^{-t\mathrm{Tr}D}\exp\left(t\sum_{\alpha\in\mathcal{G}^{(1)}} D_{\alpha\alpha}\alpha\tilde{\alpha}\right) = \bigotimes_{i=1}^n \hat{\mathcal{E}}_0^{(i)}(t) \equiv \mathrm{e}^{t\hat{\Pi}_0} \tag{6}$$

with the error operator for the ith qubit, $\hat{\mathcal{E}}_0^{(i)}(t) = \exp\left[t\sum_{\alpha_i=X_i,Y_i,Z_i} D_{\alpha_i\alpha_i}(\alpha_i\tilde{\alpha}_i - 1)\right]$. When interactions among the qubits on different sites are absent, we see that the error operator reduces to a product of $\hat{\mathcal{E}}_0^{(i)}(t)$. Note that $\hat{\mathcal{E}}_0^{(i)}(t)$ gives the depolarizing channel [1], up to $O(t)$.

Now, let us expand $\hat{\mathcal{E}}_0(t)$ and $\hat{\mathcal{E}}(t)$ up to the first order with respect to tD, i.e.,

$$\hat{\mathcal{E}}_0(t) = (1 - t\,\mathrm{Tr}\,D)I\tilde{I} + t\sum_{\alpha\in\mathcal{G}^{(1)}} D_{\alpha\alpha}\alpha\tilde{\alpha}, \tag{7}$$

$$\hat{\mathcal{E}}(t) = I\tilde{I} + \frac{t}{2}\sum_{\alpha,\beta\in\mathcal{G}^{(1)}} D_{\alpha\beta}^*\hat{\Pi}_{\alpha\beta} \equiv J\tilde{J} + t\sum_{\alpha\in\mathcal{G}^{(1)}} d_\alpha F_\alpha\tilde{F}_\alpha, \tag{8}$$

where $J = I - \frac{t}{2}\sum_{\alpha,\beta\in\mathcal{G}^{(1)}} D_{\alpha\beta}\alpha\beta$ and $F_\alpha = \sum_{\beta\in\mathcal{G}^{(1)}} U_{\alpha\beta}\beta$ with U being a unitary matrix which diagonalizes D, i.e., $(UDU^\dagger)_{\alpha\beta} = d_\alpha\delta_{\alpha\beta}$. Here, d_α are the eigenvalues of D.

Let C_1 be a stabilizer code which can correct the independent error (7), and let $\mathcal{S}_1$ be its stabilizer. The quantum error correction condition for the code is given by [11]

$$\alpha, \alpha\beta \in \left(\mathcal{G} - \mathcal{N}_{\mathcal{S}_1}\right) \cup \mathcal{S}_1 \qquad (\forall \alpha, \beta \in \mathcal{G}^{(1)}) \tag{9}$$

with the Pauli group of n qubits $\mathcal{G} = \{\pm I, \pm X, \pm \mathrm{i}Y, \pm Z\}^{\otimes n}$ and the commutant $\mathcal{N}_{\mathcal{S}_1}$ of $\mathcal{S}_1$. Note that $\mathcal{S}_1 \subset \mathcal{N}_{\mathcal{S}_1} \subset \mathcal{G}$. The error correction operation is described by a recovery operator $\hat{\mathcal{R}}_1 = \sum_{\boldsymbol{x}\in\{0,1\}^{\otimes s_1}} g_{\boldsymbol{x}}^\dagger \tilde{g}_{\boldsymbol{x}}^\dagger P_{\boldsymbol{x}} \tilde{P}_{\boldsymbol{x}}$ which satisfies $\hat{\mathcal{R}}_1 \hat{\mathcal{E}}_0(t) P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} = P_{\mathbf{0}} \tilde{P}_{\mathbf{0}}$. Here, $s_1 \equiv \dim \mathcal{S}_1$, and $P_{\boldsymbol{x}} = \prod_{a=1}^{s_1} \frac{1+(-1)^{x_a} M_a}{2}$ is the projection operator onto the simultaneous eigenspace of generators $\{M_a\}_{a=1}^{s_1}$ of $\mathcal{S}_1$ with eigenvalues $\boldsymbol{x} = (x_1, \ldots, x_{s_1}) (x_a \in \{0, 1\})$. Especially, $P_{\mathbf{0}}$ is a projection operator onto C_1. They satisfy the orthogonality and completeness conditions, i.e., $P_{\boldsymbol{x}} P_{\boldsymbol{x}'} = \delta_{\boldsymbol{x}\boldsymbol{x}'} P_{\boldsymbol{x}}$ and $\sum_{\boldsymbol{x}} P_{\boldsymbol{x}} = I$. By definition, we see that

$$E P_{\mathbf{0}} E^\dagger = P_{\boldsymbol{x}(E)} \qquad \text{for} \quad \forall E \in \mathcal{G}, \tag{10}$$

where $\boldsymbol{x}(E)$ is the syndrome of E, i.e., $E M_a E^\dagger = (-1)^{x_a(E)} M_a$. The syndrome has the property

$$\boldsymbol{x}(E) = \boldsymbol{x}(E') \Leftrightarrow \boldsymbol{x}(EE') = \mathbf{0} \qquad \text{for} \quad \forall E, E' \in \mathcal{G}. \tag{11}$$

Note also that $\boldsymbol{x}(E) = \mathbf{0} \Leftrightarrow E \in \mathcal{N}_{\mathcal{S}_1}$ for $\forall E \in \mathcal{G}$. Therefore, with the help of (9), we see that

$$\alpha\beta \in \mathcal{S}_1 \qquad \text{for} \quad \boldsymbol{x}(\alpha) = \boldsymbol{x}(\beta) \quad \text{or} \quad \boldsymbol{x}(\alpha\beta) = \mathbf{0}. \tag{12}$$

$\forall g (\in \mathcal{S}_1)$ can be expanded by the projection operators in the form $g = \sum_{\boldsymbol{x}} (-1)^{\boldsymbol{y}(g)\cdot\boldsymbol{x}} P_{\boldsymbol{x}}$, where $\boldsymbol{y}(g) = (y_1(g), \ldots, y_{s_1}(g))$ $(y_a(\bullet) \in \{0, 1\})$ is defined by $g = M_1^{y_1(g)} \cdots M_{s_1}^{y_{s_1}(g)}$. This expression implies

$$g P_{\mathbf{0}} = P_{\mathbf{0}} \qquad \text{for} \quad \forall g \in \mathcal{S}_1. \tag{13}$$

Operators $g_{\boldsymbol{x}}$ in $\hat{\mathcal{R}}_1$ are determined by the following algorithm:

`if` $\boldsymbol{x}' = \mathbf{0}$ `then` $g_{\boldsymbol{x}'} = I$

`else if` $\boldsymbol{x}' = \boldsymbol{x}(\alpha)$ `then` $g_{\boldsymbol{x}'} =$ `(one out of` $\{\beta | \boldsymbol{x}(\beta) = \boldsymbol{x}(\alpha)\}$`).`

Here, 'one out of' means that any one out of the set is equally eligible for $g_{\boldsymbol{x}'}$.

Here, it may be worthwhile noting that $\hat{\mathcal{E}}(t)$ includes the error operator J which is not a linear combination of $\{I, \alpha \in \mathcal{G}^{(1)}\}$ (see (8)). This is due to $\langle\langle |\mathrm{d}\mathfrak{B}_t^{\alpha_i} \mathrm{d}\mathfrak{B}_t^{\beta_j}| \rangle\rangle = D_{\alpha_i\beta_j} \mathrm{d}t \neq 0 (i \neq j)$ which manifests the existence of noise that has a spatial correlation between different sites i and j. In the following, we will show that $(\mathsf{C}_1, \hat{\mathcal{R}}_1)$ can correct also the spatially correlated error $\hat{\mathcal{E}}(t)$ even if it includes such 2-qubit error operator J; This is an extension of a known theorem [1] which reads as follows. *When a code can correct the depolarizing channel* $\hat{\mathcal{E}}_D = p_0 I\tilde{I} + p_1 X\tilde{X} + p_2 Y\tilde{Y} + p_3 Z\tilde{Z}$, *it also corrects any 1-qubit error, i.e.,* $\hat{\mathcal{E}} = \sum_a E_a \tilde{E}_a$ *with each* E_a *being a linear combination of* $\{I, X, Y, Z\}$ *and satisfying* $\sum_a E_a^\dagger E_a = I$.

For $\forall \alpha, \beta \in \mathcal{G}^{(1)}$, one can show that

$$\begin{aligned} \hat{\mathcal{R}}_1 \alpha\tilde{\beta} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} &= \hat{\mathcal{R}}_1 P_{\boldsymbol{x}(\alpha)} \tilde{P}_{\boldsymbol{x}(\beta)} \alpha\tilde{\beta} = \delta_{\boldsymbol{x}(\alpha)\boldsymbol{x}(\beta)} g_{\boldsymbol{x}(\alpha)}^\dagger \tilde{g}_{\boldsymbol{x}(\beta)}^\dagger P_{\boldsymbol{x}(\alpha)} \tilde{P}_{\boldsymbol{x}(\beta)} \alpha\tilde{\beta} \\ &= \delta_{\boldsymbol{x}(\alpha)\boldsymbol{x}(\beta)} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} \underbrace{g_{\boldsymbol{x}(\alpha)}^\dagger \alpha}_{\in \mathcal{S}_1} \underbrace{\tilde{g}_{\boldsymbol{x}(\beta)}^\dagger \tilde{\beta}}_{\in \tilde{\mathcal{S}}_1} = \delta_{\boldsymbol{x}(\alpha)\boldsymbol{x}(\beta)} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} = \delta_{\boldsymbol{x}(\alpha\beta)\mathbf{0}} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}}. \end{aligned} \tag{14}$$

Here, we have used (10) at the first equality, the orthogonality of $P_{\boldsymbol{x}}$'s at the second, (10) again at the third, (12) and (13) at the fourth, and (11) at the last equality. Similarly,

$$\hat{\mathcal{R}}_1 \alpha\beta P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} = \hat{\mathcal{R}}_1 P_{\boldsymbol{x}(\alpha\beta)} \tilde{P}_{\mathbf{0}} \alpha\beta = \delta_{\boldsymbol{x}(\alpha\beta)\mathbf{0}} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} \alpha\beta = \delta_{\boldsymbol{x}(\alpha\beta)\mathbf{0}} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}}. \tag{15}$$

With (14), (15) and its tilde-conjugate, we have

$$\hat{\mathcal{R}}_1 \hat{\Pi}_{\alpha\beta} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} = (2 - 1 - 1) \delta_{\boldsymbol{x}(\alpha\beta)\mathbf{0}} P_{\mathbf{0}} \tilde{P}_{\mathbf{0}} = 0. \tag{16}$$

One can observe that the cancellation in (16) is due to the combination of coefficients, i.e., $(2, -1, -1)$, in $\hat{\Pi}_{\alpha\beta}$. Note that this combination has been required by $\langle\!\langle\theta|\hat{\Pi} = 0$, i.e., conservation of probability within the relevant system.

Now let us proceed to mth order of expansion of $\hat{\mathcal{E}}_0(t)$ and $\hat{\mathcal{E}}(t)$. Let C_m and $\mathcal{S}_m$, respectively, be a stabilizer code which can correct $\hat{\mathcal{E}}_0(t)$ up to $O\big((t\hat{\Pi}_0)^m\big)$ and its stabilizer. The quantum error correction condition reads

$$\alpha_1, \alpha_1\alpha_2, \ldots, \alpha_1 \ldots \alpha_{2m} \in \left(\mathcal{G} - \mathcal{N}_{\mathcal{S}_m}\right) \cup \mathcal{S}_m \tag{17}$$

$(\forall\alpha_i \in \mathcal{G}^{(1)})$. The corresponding recovery operator $\hat{\mathcal{R}}_m = \sum_{\boldsymbol{x}\in\{0,1\}^{\otimes s_m}} g_{\boldsymbol{x}}^\dagger \tilde{g}_{\boldsymbol{x}}^\dagger P_{\boldsymbol{x}} \tilde{P}_{\boldsymbol{x}}$ ($s_m = \dim \mathcal{S}_m$) is determined by the following algorithm:

`if` $\boldsymbol{x}' = \boldsymbol{0}$ `then` $g_{\boldsymbol{x}'} = I$

`else if` $\boldsymbol{x}' = \boldsymbol{x}(\alpha_1)$ `then` $g_{\boldsymbol{x}'} =$ `(one out of` $\{\beta_1|\boldsymbol{x}(\beta_1) = \boldsymbol{x}(\alpha_1)\}$`)`

$\vdots$

`else if` $\boldsymbol{x}' = \boldsymbol{x}(\alpha_1 \cdots \alpha_m)$ `then`

$g_{\boldsymbol{x}'} =$ `(one out of` $\{\beta_1 \cdots \beta_m|\boldsymbol{x}(\beta_1 \cdots \beta_m) = \boldsymbol{x}(\alpha_1 \cdots \alpha_m)\}$`)`.

Note that $P_{\boldsymbol{0}}$ here is the projection operator onto C_m.

It is easy to observe that $\hat{\Pi}^m$ is a linear combination of products of the forms $\alpha_1 \cdots \alpha_p\tilde{\alpha}_{p+1} \cdots \tilde{\alpha}_{2m} (1 \leqslant p \leqslant m), \alpha_1 \cdots \alpha_{2m}$ and their tilde-conjugates, where $\alpha_i \in \mathcal{G}^{(1)}$ and $\tilde{\alpha}_i \in \tilde{\mathcal{G}}^{(1)}$. For each of these products, we see that

$$\begin{aligned}
&\hat{\mathcal{R}}_m\alpha_1 \cdots \alpha_p\tilde{\alpha}_{p+1} \cdots \tilde{\alpha}_{2m} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}} \\
&\quad = \hat{\mathcal{R}}_m P_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)} \tilde{P}_{\boldsymbol{x}(\alpha_{p+1}\cdots\alpha_{2m})}\alpha_1 \cdots \alpha_p\tilde{\alpha}_{p+1} \cdots \tilde{\alpha}_{2m} \\
&\quad = \delta_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)\boldsymbol{x}(\alpha_{p+1}\cdots\alpha_{2m})} g^\dagger_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)} \tilde{g}^\dagger_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)} P_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)} \tilde{P}_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)}\alpha_1 \cdots \alpha_p\tilde{\alpha}_{p+1} \cdots \tilde{\alpha}_{2m} \\
&\quad = \delta_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)\boldsymbol{x}(\alpha_{p+1}\cdots\alpha_{2m})} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}} \underbrace{g^\dagger_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)}\alpha_1 \cdots \alpha_p}_{\in\mathcal{S}_m} \underbrace{\tilde{g}^\dagger_{\boldsymbol{x}(\alpha_1\cdots\alpha_p)}\tilde{\alpha}_{p+1} \cdots \tilde{\alpha}_{2m}}_{\in\tilde{\mathcal{S}}_m} \\
&\quad = \delta_{\boldsymbol{x}(\alpha_1\cdots\alpha_{2m})\boldsymbol{0}} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}}
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
&\hat{\mathcal{R}}_m\alpha_1 \cdots \alpha_{2m} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}} = \hat{\mathcal{R}}_m P_{\boldsymbol{x}(\alpha_1\cdots\alpha_{2m})} \tilde{P}_{\boldsymbol{0}}\, \alpha_1 \cdots \alpha_{2m} \\
&\quad = \delta_{\boldsymbol{x}(\alpha_1\cdots\alpha_{2m})\boldsymbol{0}} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}}\alpha_1 \cdots \alpha_{2m} = \delta_{\boldsymbol{x}(\alpha_1\cdots\alpha_{2m})\boldsymbol{0}} P_{\boldsymbol{0}} \tilde{P}_{\boldsymbol{0}}.
\end{aligned} \tag{19}$$

Therefore, we have $\hat{\mathcal{R}}_m\hat{\Pi}^m P_{\boldsymbol{0}}\tilde{P}_{\boldsymbol{0}} = cP_{\boldsymbol{0}}\tilde{P}_{\boldsymbol{0}}$ with c being a c-number. Taking an expectation value of this equation with respect to the thermal vacua, we see that $c\langle\!\langle\theta|P_{\boldsymbol{0}}|0\rangle\!\rangle = 0$ since $\langle\!\langle\theta|\hat{\mathcal{R}}_m = \langle\!\langle\theta|$, $\langle\!\langle\theta|\hat{\Pi} = 0$ and $\langle\!\langle\theta|\tilde{P}_{\boldsymbol{0}} = \langle\!\langle\theta|P_{\boldsymbol{0}}$. This implies $c = 0$ since $|0\rangle\!\rangle$ can be any eligible initial ket-vacuum. Then, we have

$$\hat{\mathcal{R}}_m\hat{\Pi}^m P_{\boldsymbol{0}}\tilde{P}_{\boldsymbol{0}} = 0. \tag{20}$$

Similar results hold for $\hat{\Pi}^l$ with $1 \leqslant l < m$. Thus, we obtain

$$\hat{\mathcal{R}}_m\hat{\mathcal{E}}(t) P_{\boldsymbol{0}}\tilde{P}_{\boldsymbol{0}} = P_{\boldsymbol{0}}\tilde{P}_{\boldsymbol{0}} \tag{21}$$

up to $O((t\hat{\Pi})^m)$.

Let us summarize the results. The spatial correlation of noise alters the error operator into one which is not covered by the aforementioned theorem (see (7) and (8)). However, within the present model of spatially correlated noise, we have shown that the changes are something which can be dealt with, at each order, by the stabilizer code prepared for independent errors. In other words, we have confirmed that the spatial correlation of noise do not cause any extra errors beyond the conventional stabilizer code. Since the present noise model is general enough, we expect this concept is valid through broad situations. Here, it is worthwhile noting that although J in (8) includes 2-qubit operator $\alpha_i \beta_j$, it does not actually bring in '2-qubit error' up to $O(tD)$ since the term $\frac{t^2}{4} \sum_{\alpha,\beta,\alpha',\beta' \in \mathcal{G}^{(1)}} D_{\alpha\beta} D^*_{\alpha'\beta'} \alpha\beta\tilde{\alpha}'\tilde{\beta}'$ is of $O((tD)^2)$. This is one of the reasons why the cancellation (16) occurs.

As we have shown that $\hat{\mathcal{E}}(t)$ is corrected by the existing m-error correcting stabilizer code up to $O((t\hat{\Pi})^m)$, the spatial correlation of noise has been confirmed to figure in only the enhancement of error rate at $O((t\hat{\Pi})^{m+1})$. Note that the preceding study [28] accounts for the case of 1-error correcting non-degenerate code for a phase-flip error only. Our result is a much further extended one since it is valid for any m-error correcting stabilizer code (degenerate or non-degenerate) with $m \geqslant 1$. Similar but limited results are found in [29] and in [30]. In the former, it is shown that spatially correlated errors can be corrected in the cases of n-qubit repetition code and two CSS codes, i.e., the $[\![7, 1, 3]\!]$ Hamming code and $[\![23, 1, 7]\!]$ Golay code. In the latter, a similar result as ours is found for the case of general CSS code. Note also that all of these preceding studies are dealt with the help of state fidelity $F = \langle\!\langle 0|\hat{\mathcal{R}}\hat{\mathcal{E}}(t)|0\rangle\!\rangle$, i.e., an expectation value. Here, $\langle\!\langle 0| = |0\rangle\!\rangle^\dagger$. In contrast, our present result (21), an operator equation, is derived with the help of operator algebra within NETFD.

In this paper, we did not consider the effects of temporal correlation of noises, i.e., we assumed that the noise is white. In order to deal with temporal noise correlations within our present approach, we have to extend our framework of stochastic differential equations to allow non-white noise; this is an attractive future problem. One can have, however, a considerable outlook about the effect of temporal noise correlation by analysing an exactly solvable model of decoherence, i.e., the *spin–boson decoherence model* [8, 31–33], where only phase-flip errors are considered. The error operator $\hat{\mathcal{E}}_{\mathrm{SBD}}(t)$ for the model is obtained exactly (i.e., without white-noise approximations). The result reads

$$\hat{\mathcal{E}}_{\mathrm{SBD}}(t) = \exp\left\{\frac{1}{2}\sum_{i,j=1}^{n}[\lambda_{ij}(t)(2Z_i\tilde{Z}_j - Z_jZ_i - \tilde{Z}_i\tilde{Z}_j) + 2\mathrm{i}\theta_{ij}(t)(Z_jZ_i - \tilde{Z}_i\tilde{Z}_j)]\right\} \tag{22}$$

with analytical functions $\lambda_{ij}(t)$ and $\theta_{ij}(t)$. Expanding $\hat{\mathcal{E}}_{\mathrm{SBD}}(t)$ in terms of $\ln\hat{\mathcal{E}}_{\mathrm{SBD}}(t)$, one can observe that a similar mechanism as (20) works for $(\ln\hat{\mathcal{E}}_{\mathrm{SBD}}(t))^m$ $(m \geqslant 1)$ since $\langle\!\langle\theta|\ln\hat{\mathcal{E}}_{\mathrm{SBD}}(t) = 0$. With these observations, we conjecture that *spatially and/or temporally correlated* errors can be corrected by the same stabilizer code for spatially and temporally independent error $\hat{\mathcal{E}}_0(t) = \mathrm{e}^{t\hat{\Pi}_0}$, at each order of expansion in terms of $t\hat{\Pi}_0$.

Recently, a method to construct *non-stabilizer Clifford codes* has been developed [34]. It is another interesting challenge to extend our present study to non-stabilizer Clifford codes.

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